Assignment 5.

Basics of complex integral. Random stuff

This assignment is due Wednesday, Feb 20. Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

(1) Compute the Riemann sums and corresponding limits in the following cases.
  (a) $f(z) = 1$, $\gamma : [a, b] \to \mathbb{C}$ arbitrary s.t. $\gamma(a) = z_0$, $\gamma(b) = Z$, $\dot{\mathcal{P}}$ arbitrary. Find $S(f, \dot{\mathcal{P}})$ and its limit as $|\dot{\mathcal{P}}| \to 0$.
  (b) $f(z) = z$, γ as above, $\dot{\mathcal{P}}_1$ s.t. $\tau_k = t_k$, $\dot{\mathcal{P}}_2$ s.t. $\tau_k = t_{k-1}$. Find $\frac{1}{2}(S(f, \dot{\mathcal{P}}_1) + S(f, \dot{\mathcal{P}}_2))$. Find limit of this expression as $|\dot{\mathcal{P}}_{1,2}| \to 0$.
  (c) $f(z) = 1/z$, $\gamma : [0, 2\pi] \to \mathbb{C}$ a circle $\gamma(t) = re^{it}$. Partition $\dot{\mathcal{P}}$ as follows: t_k's divide circle into equal arcs, $t_k = 2k\pi/n$ $(k = 0, 1 \ldots, n)$; τ_k are the midpoints of corresponding arcs, $\tau_k = (2k-1)\pi/n$ $(k = 1, \ldots, n)$. Find $S(f, \dot{\mathcal{P}})$ and its limit as $|\dot{\mathcal{P}}| \to 0$.

  Comment. The functions above are continuous on the corresponding paths, so we know from calculus that the functions are integrable. Therefore, the limits you found above actually equal to the values of the respective integrals.

  Comment. You also might have noticed that, just as in case of real valued Riemann integral, finding integral through Riemann sums is inconvenient.

(2) Find the integral $\int_\gamma f(z)dz$ for f and γ in the item 1c using the formula $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$. (*Hint:* If the answer is different from what you got before, something is wrong.)

Problems below don't necessarily have anything to do with the integral. Rather, they highlight some differences between real and complex numbers.

(3) Prove that there does not exist a subset P of $\mathbb{C}$ with the following properties:
  - For every $z \in \mathbb{C}$, exactly one of the following holds: $z \in P$, or $-z \in P$, or $z = 0$.
  - For every $z, w \in P$, the sum is also in P: $z + w \in P$.
  - For every $z, w \in P$, the product is also in P: $zw \in P$.

  Comment. This explains that $<$ and $>$ cannot be defined on entire $\mathbb{C}$ consistently with arithmetic operations. If they were, then the set $\{z > 0\}$ would work as such P. (P stands for *positive*, by the way.)

(4) Give counterexample to Mean Value Theorem for complex numbers. For instance, you can find a function $f : \mathbb{C} \to \mathbb{C}$ s.t. $f(z_1) = f(z_2)$ for some $z_1 \neq z_2$, but f' is never 0. (*Hint:* What functions with nowhere zero derivative do you know?)

(5) Prove that if two functions $f, g : G \to \mathbb{C}$ defined on a domain $G \subseteq \mathbb{C}$ are differentiable on G and $f' = g'$, then $f = g + C$ for some constant C. (*Hint:* Note that in real analysis, this is proved using the mean value theorem which fails for complex numbers as seen above. Here, instead, use expression of f' through partial derivatives.)

(6) Compute $\int_\gamma e^{2\pi i z}dz$, where γ is the interval $[0, 1]$ of real line. Observe that the Mean Value Theorem for integral fails, too.